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DECEMBER 1967

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Technical Memorandum

STATISTICAL ANALYSIS OF RADAR TARGET SCINTILLATION

by E. SHOTLAND

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AFID, Washington, D.C. 20301

TG-955

DECEMBER 1967

Technical Memorandum

**STATISTICAL ANALYSIS
OF RADAR TARGET SCINTILLATION**

by E. SHOTLAND

SPONSORED BY ARPA UNDER AO #479

THE JOHNS HOPKINS UNIVERSITY ■ APPLIED PHYSICS LABORATORY
8621 Georgia Avenue, Silver Spring, Maryland 20910

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ABSTRACT

A new theoretical approach to the problem of radar-target scintillation was developed. It was applied to slender, axially symmetrical targets that are much longer than the RF wave length. The radar scintillation is analyzed and described in terms of statistical parameters. The following sets of quantities were derived and computed:

- (a) mean radar cross sections, their RMS fluctuations and average lobing frequencies;
- (b) mean target centroids, their RMS deviations and average meandering rates.

These and other variables were expressed as functions of the aspect angle, radio frequency and of configurational details.

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I. INTRODUCTION TO THE PROBLEM OF RADAR-TARGET SCINTILLATION

The radar characteristic known as "target scintillation" was defined by Muchmore [1] as a physical phenomenon that consists of two effects: (1) the fluctuation of the rectified amplitude of the target echo as observed in the video circuits, and (2) the wander of the apparent centroid of target position as generated by the output of a tracking circuit.* The scintillation of the stars also consists of two effects: fluctuation of magnitude and variation of position. However, it was pointed out by Muchmore that the causes of stellar and radar target scintillations are distinct. Star light twinkle is caused by the disturbances of the intervening medium (atmosphere) whereas radar aircraft fluctuation is produced by the interference between many scattering points of a usually complex target. It will be shown later that the properties of a radar receiver also may affect the character of radar scintillation.**

For the design of an efficient radar and weapons system, it is important to obtain a thorough understanding of the analytical

*In the present report we are concerned with target-centroid wander in the angular space of the radar field of view. However, similar methods may also be applied to circuits producing radar range and radial range rate (Doppler). We shall follow Muchmore's definition of "target scintillation" although some authors deviate from this nomenclature. For instance Barton (Ref. [27]) refers to "amplitude noise" as scintillation which other authors call "fading noise." The term "glint" is generally applied to "angular noise," the quantity "radar jitter" is a collective term that also includes the effects of receiver and servo noise.

**See Appendix, Secs. A.IV and A.V.3.

and statistical character of the target echo. The latter enters into two phases of radar operation. During the stage of target detection and acquisition, the mean radar cross section (abbreviated "RCS" in the sequel), its fluctuation and time rate of fading play an important role. The success of this operation can be measured and computed by means of tables and graphs of pertinent references (cf. [2], [3], [4]). An important entry of these tables is the "category of the target," which is specified by several parameters. One important characteristic is the average duration of the fading interval as compared to the pulse repetition and scanning periods of the radar.

During the target tracking and missile semi-active homing stages of a missile-target intercept, similar problems relative to radar performance are involved. The RMS fluctuation of the mean target centroid and its rate of oscillatory meander are both significant and somewhat parallel in concept. The problem of "glint" caused by the erratic wandering of the centroid, especially for extended targets at close ranges, is akin to the multiple-target problem which has not yet been solved in a satisfactory manner.

The knowledge of hostile target scintillations can be obtained (a) by actual flight observations, (b) by static radar-ground tests carried out on models and (c) by analysis and computation, provided the configuration of the enemy craft is known from photos or other means of intelligence. The present analysis belongs to the latter category. It offers the following

advantages: (1) cost reduction by avoiding hardware and test operations during preliminary design stages of the defensive weapon system, (2) flexibility in the choice of systems parameters, and (3) mathematical evaluation of possible schemes for dealing with heavily fluctuating targets. These schemes include the selection of polarization, radio frequency modulation, narrow-band filters the center frequencies of which can be adapted to measured instantaneous frequencies, adjustable automatic gain controls, etc.

II. BACKGROUND OF PAST AND PRESENT RESEARCH ON THE RADAR SCINTILLATION PROBLEM

Various distinct approaches to the problem of radar target scintillation can be found in the literature. The experts in the field of radar signatures (K. M. Siegel and collaborators, see Ref. [5]), developed an extensive physical and mathematical theory by means of which the RCS of a complex structure (consisting of hundreds of stations) can be computed as a function of aspect angle, frequency and polarization. Good agreement between analytical results and measured radar echoes is claimed. Some of the underlying assumptions and mathematical principles will be discussed in Appendix A.I. The statistical results of this research team are limited to concepts of so-called "first-generation statistics," such as $\bar{\sigma}$ (mean cross section), S (standard deviation of σ) etc. No use is made of quantities appearing in "higher-dynasty statistics" such as multiple joint distributions, correlation functions and spectra.

A different type of a statistical theory of aircraft scintillation was developed by Delano [6] and Muchmore [1]. Their results appear in terms of spectra and probability densities. However, their theory applies to simple airframe structures that can be considered as an assembly of a few individual scatterers or of a few simple areas of uniformly distributed scatterers. Related problems were analyzed in Refs. [7], [20], [21], and [22]. Some interesting, statistical work was carried out by Robert W. Kennedy, Major USAF. He was primarily concerned with radar cross sections of simple, satellite-type targets. (See Ref. [8]).

A natural extension of the above-cited work would call for a combination of the ideas incorporated separately in the investigations of the aforementioned research teams. It would then yield, for instance, the RCS spectra of complicated target configurations. The preparation of such quantities presents a

cumbersome computational chore but it can be accomplished. However, the results would still be inadequate for the design of radars against specific complex targets, for the following two reasons:

(1) "Spectra," in a strict sense, are only applicable to stationary processes. They furnish the mean square amplitude densities in the frequency domain, averaged over an infinite interval of time. The radar operators in the field, however, are well aware of the fact that the signatures of slender targets vary markedly in magnitude and spectral content when turning from broadside to frontal aspect. What is needed here, is the knowledge of temporary signal behavior. The concept and the analysis of "locally stationary processes" will be introduced and utilized in the present report to cope with this phase of the problem. (See Appendix, Sec. A.II).

(2) Often one deals with complex targets consisting of many component scatterers that are somewhat randomly distributed over the length of the body. Radar interference between these scatterers generates many nulls which appear randomly distributed in time.* In the extreme case of a Poisson distribution, these lobes generate a flat spectrum according to Ref. [9]. This very fact lies at the heart of the multiple-target problem. If one deals with narrow-band targets in the background of wide-band noise, one can design narrow-band filters that emphasize the target signals and wash out the noise. If one deals with wide-band target signals,** no ordinary filter is helpful. If one washes out the noise one filters the target signal as well. The physical implication of this fact is well known to the radar operator in the field. If he tracks two targets of equal strength appearing simultaneously and a fraction of a beam width apart, the radar axis will wander erratically from one target to the other and sometimes stray

* even if the target turns at constant angular rate.

** In many references "target signals" are separated into two parts: (1) "true target signals" such as "true angular positions" and (2) "target noise" such as "glint." For other quantities such as the "RCS" σ , this separation is not as simple. For some variables of closed-loop missile guidance, this separation becomes even arbitrary and depending on the choice of the analyst.

beyond these points until the signal gets lost. If the receiver applies a narrow band tracking filter or a slower antenna servo, the radar axis still undergoes excursions of the same magnitude, only at a slower rate. Apparently what is needed here, in order to cope with this difficulty, is a better knowledge of the so-called "instantaneous frequency" and of other parameters developed by S. O. Rice under the heading of "Statistics of the Count of Zero-crossings." (Ref. [10]). J. W. Follin, Jr., of APL proposed to apply this segment of Rice's random noise analysis to the problem of radar-target scintillation. The present author performed the mathematical analysis connected with this problem and the Conductron Corporation carried out the numerical computations, (cf. References [11], [12], [13], [15], [19]).

The major results of this investigation will appear in Section VII. For our specific task a particular type of target configuration was chosen and used in our numerical calculations. Tactical geometry and the choice of target coordinates will be explained in Section III. General definitions and symbols are given in Section IV, specific definitions and refined statistical notation in Section V. The underlying assumptions which primarily pertain to our special project, are compiled in Section VI. In addition, we used a number of general mathematical principles which were very helpful in rendering the computational chores tractable. In order not to encumber the main stream of information contained in the major body of this report, mathematical principles and pertinent theorems were relegated to special sections of the appendix, as were refined details of the mathematical formulation of the problem and of the derivation of our formulae.

III. TACTICAL GEOMETRY AND TARGET COORDINATES

The tactical geometry and target coordinates are explained in Figure 1. Two cases must be considered:

(1) Monostatic Case

This case refers to the operation of a tracking radar and the action of an actively homing missile. The radar-line of sight and the target-longitudinal axis form a plane (P) which is depicted in Figure 1. Of special interest are vehicles with axial symmetry. For this special configuration the RCS problem becomes two-dimensional since one spatial dimension (Z-coordinate) can be suppressed.

The choice of the coordinate system is shown in Figure 1. The origin coincides with the nose tip (station "0"), the x-axis with the longitudinal axis of the target, and the y-axis lies in plane P. The vehicles are divided into N sections whose end points on the contour or some other convenient locations are selected as the stations S_1 which represent the individual component scatterers.

More details which are important in the analysis are sketched in Figure 2.

Besides the stations S_1 , the positions of the instantaneous centroid S and of the mean centroid \bar{S} are important. In References [1] and [6], the latter are designated as the "apparent" and "effective" radar centers respectively. It is convenient to refer the so-called error angles ϵ_1 to the line of sight toward \bar{S} . Then one has $\bar{\epsilon} = 0$. If the radar axis is stabilized toward \bar{S} , the error voltage, averaged over a long period, also vanishes. The aspect angle $\bar{\theta}$ is the angle between the x-axis and the LOS from the tracker T to \bar{S} . The direction of the LOS is given by the unit vector \vec{U}_0 , the direction at right angles to it by the unit vector \vec{U}_ν . Note that TS and TS_1 are almost parallel and the angle ϵ_1 is very small under "far-field conditions." The remaining quantities are indicated in Figure 1, and are self-explanatory.

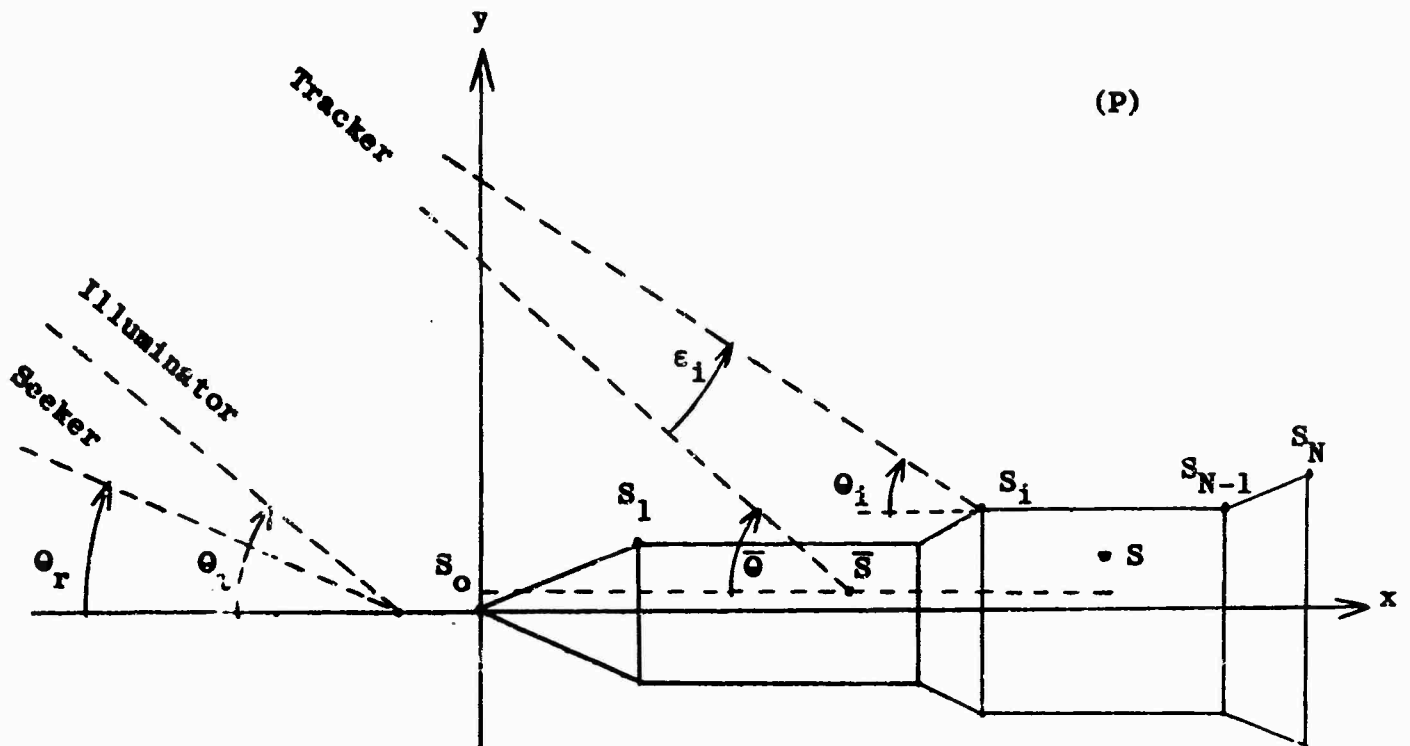


Figure 1. Tactical Geometry and Target Coordinates

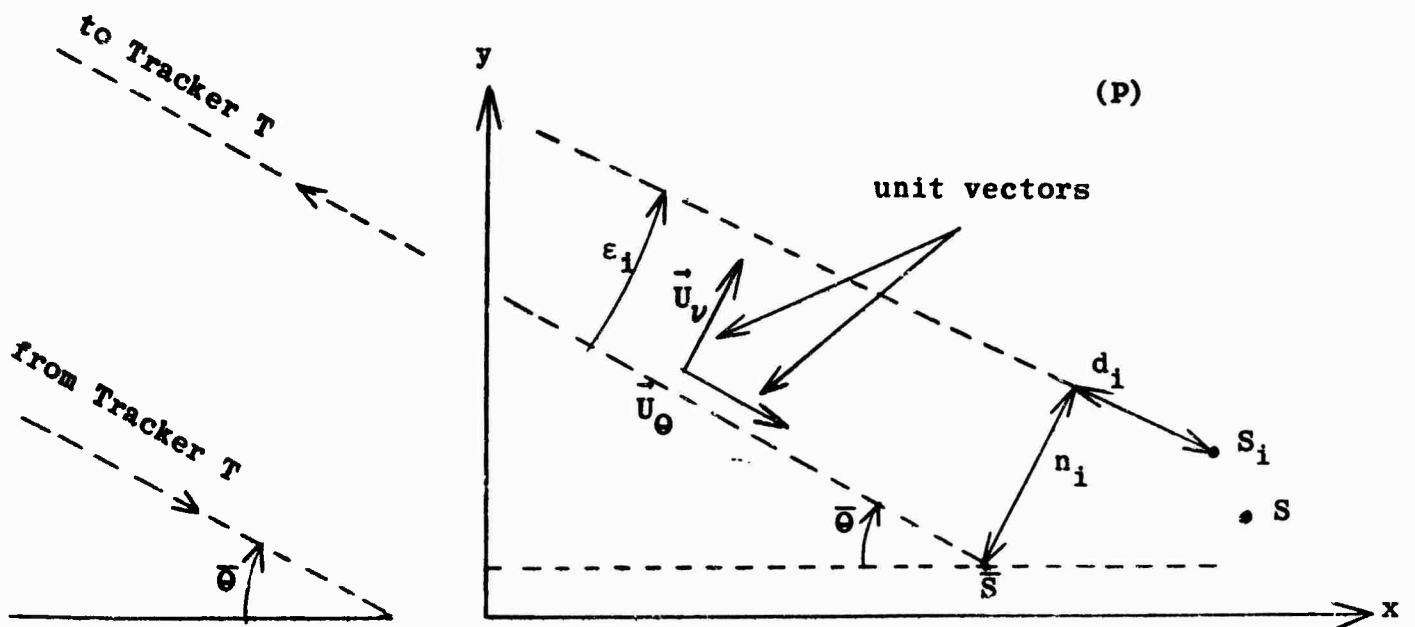


Figure 2. Details of Geometry

(2) Bistatic Case

This case refers to the action of a semi-active homing interceptor. The target configuration is the same. However, the single aspect angle θ must be replaced by two angles, θ_t and θ_r . The transmitter angle θ_t is the angle between the illuminator LOS and the x-axis; the receiver angle θ_r is the angle between the seeker LOS and the x-axis, as indicated in Figure 1. In the definition of the error angles ϵ_i , the LOS from the seeker to the mean centroid \bar{S} forms the basic reference line in the bistatic case, since it is the seeker receiver that measures the error voltage.

IV. GENERAL DEFINITIONS AND SYMBOLS

In the analysis of radar cross sections as applied to the theory of target detection, the following quantities play an important role:

σ is the radar cross section of a body, in m^2 .
The RCS is 4π times the re-radiated electromagnetic power, in watts per steradian, divided by the irradiated power per unit area, in watts per m^2 .

σ_p is the instantaneous RCS of a body computed by the method of "relative phase." Where there is no confusion likely, we shall simply omit the subscript "p".

σ_i is the RCS of the individual component scatterer.

$\bar{\sigma}$ is the mean RCS (Ordinarily, this is a time average. $\bar{\sigma} = \sigma(t)_{av}$. However, σ might also be considered as a parameter process with the aspect angle θ or the frequency f representing the independent parameter. The mean then is taken with respect to the parameter. *)

$\Delta\sigma = \sigma - \bar{\sigma}$ is the instantaneous deviation of the RCS from its mean.

$\sigma_{sd} = (\Delta\sigma^2)^{1/2}$ is the standard deviation of the RCS.
(Note: Reference [5] uses the symbol "S".)

*In the bistatic case one has $\sigma = \sigma \{\theta_t, \theta_r\}$ and the average can be defined as $\bar{\sigma} = \sigma \{\theta_t(q), \theta_r(q)\}_{av}$ where the averaging operation is extended over the parameter q .

σ_{MAX} is the maximum possible RCS (taken over all possible RF-phases for fixed parameters).

In the analysis of radar target glint as applied to the theory of target tracking, the following definitions and symbols are important:

(The various stations of the target are indicated by vectors or matrix columns.* Vectors are denoted by capitals, their components by lower-case letters. Conventional matrix symbolism is used; in particular, primes designate transposed matrices.)

$X_1 = [x_1, y_1]' = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ - position of station S_1

$X = [x, y]'$ - position of instantaneous centroid S (slow AGC).

$\bar{X} = [\bar{x}, \bar{y}]'$ - position of mean centroid \bar{S} (slow AGC).

$\Delta X_1 = X_1 - \bar{X}$ - deviation of station S_1 from \bar{S}

$\Delta X = X - \bar{X}$ - deviation of instantaneous centroid from mean.

$(\Delta X \cdot \Delta X')$ - covariance matrix of centroid position

$x_{sd} = (\overline{\Delta x^2})^{1/2}$; $y_{sd} = (\overline{\Delta y^2})^{1/2}$

$U_\theta = [\cos \theta, -\sin \theta]'$ unit vector in the direction of the radar LOS. (θ - line)

$U_\nu = [\sin \theta, \cos \theta]'$ unit vector normal to the θ - line.

r - range from radar to target, in units consistent with units of x .

* In this analysis the vectors and matrices are of dimension "2".

As mentioned in Section III, it is convenient to refer the error angles ϵ to the position of the mean centroid. Hence

$$\epsilon_i = \frac{(\Delta X_i' \cdot U_V)}{r} = \frac{n_i}{r} = \text{error angle of station } S_i \quad (1a)$$

(see Figure 2)

$$\epsilon = \frac{(\Delta X' \cdot U_V)}{r} = \text{error angle of instantaneous centroid} \quad (1b)$$

and, by definition,

$$\bar{\epsilon} = 0 \quad (1c)$$

$$\epsilon_{sd} = (\bar{\epsilon^2})^{1/2} \quad (1d)$$

Simple summations extend over all scatterers S_i with $i = 1, 2, \dots, N$. We shall use the following abbreviations:

$$\sum_{i=1}^N = \sum_i$$

Double summations extend over all combinations of $i, j = 1, 2, \dots, N$ except terms with the same index:

$$\sum_{1 \leq i \neq j \leq N} = \sum_{i \neq j} = 2 \sum_{1 \leq i < j \leq N} = 2 \sum_{i < j}$$

V. SPECIAL DEFINITIONS AND SYMBOLS APPLICABLE TO RICE'S RANDOM NOISE THEORY

For the application of S.O. Rice's "Random Theory of Noise" (Reference [10]), a few statistical definitions are necessary.

Let $z = z(q)$ be a parameter process. (In our case, z will be the RCS process $\sigma(q)$, while the independent parameter q could be the time t , the aspect angle θ , or the radio frequency f .) \bar{z} is the mean of $z(q)$ averaged over q , and $\langle z \rangle$ is the ensemble mean of $z(q)$. For homogeneous, ergodic processes, one has:

$$\bar{z} = \langle z \rangle. \quad (2)$$

The RCS process is a locally homogeneous process (locally stationary* if $q = t$), and equation (2) is valid at least for a short range of the parameter q . Let

$$\Delta z = z(q) - \bar{z}.$$

Then Δz is an unbiased random process.

The quantity $N = N[\Delta z | \Delta q, q, p, \dots]$ is defined as the number of zero crossings with positive slope of a sample process $\Delta z(q)$ in an interval I ranging from q to $q + \Delta q$. N itself is a random variable that depends on $\Delta q, q$, and possibly on some other parameters, such as p, \dots ; $\bar{N} = \bar{N}[\Delta z | \Delta q; q, p, \dots]$ is the mean of N . $\Delta N = N - \bar{N}$ is the deviation of zero counts of a particular sample process from its mean value.

$$\bar{N}_q = \bar{N}_q [\Delta z; q, p, \dots] = \lim_{\Delta q \rightarrow 0} \frac{\bar{N}[\Delta z | \Delta q, q, p, \dots]}{\Delta q}$$

* See Appendix A.II.

is the mean count of zero crossings per unit of q . \bar{N}_q is a density with dimension q^{-1} . $N_{sd} = N_{sd}(\Delta q) = (\Delta N^2)^{1/2}$ is the standard deviation of counts of zero crossings with positive slope, taken over the interval I .

A properly defined standard deviation of zero crossing density (or rate of counts) would be:

$$(N_{sd})_q = \left[\lim_{\Delta q \rightarrow 0} \frac{(\Delta N(\Delta q))^2}{\Delta q} \right]^{1/2}$$

VI. BASIC ASSUMPTIONS AND PRINCIPLES USED IN THE ANALYSIS

The following basic assumptions were utilized in our analysis:

Basic Assumptions

- (1) The target is a slender, axially symmetrical vehicle,
- (2) it consists of many independent scatterers, that are randomly distributed alongside the body,
- (3) the linear dimensions of the target and of the distances between individual scatterers are large compared to the radio wavelength λ ,
- (4) shadowing effects and multiple scattering can be ignored,
- (5) the range r from the radar to the target is large compared to the target length,
- (6) the target lies within the regime of linear error patterns of the receiving antenna,
- (7) the antenna axis is space stabilized in the direction of the mean target centroid,
- (8) the receiver uses slow AGC^{*} and square-law detection,**
- (9) receiver noise and other types of interference are ignored.

Assumptions (1) through (4) are typical for the special kind of targets which we investigated, assumptions (5) to (9) pertain to the properties of antenna and receiver and to their geometrical relationship to the target. All these premises are very useful in simplifying the analytical, statistical and computational part of this study, especially if combined with a few mathematical and physical principles which will be listed below and more fully discussed in the appendix.

* See discussion at the end of Section A.IV.

** cf. remarks in Section A.V.3.

Assumption (1) enables us to reduce the problem from a 3-dimensional to a 2-dimensional job. Assumption (9) implies that receiver noise and target scintillations are statistically independent of one another and therefore can be treated separately and then combined.

Mathematical and Physical Principles which Simplify the Analysis

- (a) Far-Field Scattering,
- (b) Born-Approximation,
- (c) Method of Random, Relative Phase,
- (d) Central Limit Theorem,
- (e) Principle of Local Stationarity,
- (f) Lord Kelvin's Principle of Stationary Phase.

Items (a), (b) and (c) were effectively used by the Conductron Corporation (cf. Ref. [5]) and will be discussed in Appendix A.I. The central limit theorem (item (d)) is a consequence of assumption (2). It enables us to utilize Gaussian statistics. This property and the principle of local stationarity (item (e)) entitle us to make use of some results of Rice's random noise analysis. Principle (e) was introduced by us for this purpose. It is a consequence of assumptions (2) and (3) and will be described in Appendix A.II. Lord Kelvin's principle of stationary phase (item (f)) is very useful in simplifying the complexity of the numerical analysis. In our case, some of the final expressions consist of fourfold summations, running over all the coordinates of the individual scatterers. If we deal with 50-100 scatterers, the number of terms required for one single quantity as a function of a set of fixed parameters (say frequency, aspect angle and polarization) may tax the capability of a modern electronic computer. The principle of stationary phase enables us to reduce quadruple sums to double sums. Details of this method will be found in Appendix A.II.

VII. SUMMARY OF MATHEMATICAL RESULTS

The results will be divided into two main sections 7.1 and 7.2 entitled "Simple Statistics" and "Refined Statistics" respectively. The former contains formulae which were obtained previously by other researchers (cf. Refs. [5] and [19]), the second main section applies the methods of Rice's analysis to the scintillation problem. The results are believed to be new, to the best of our knowledge.

Each main section is divided into two subsections, e.g., 7.1.1 and 7.1.2. The former deals with quantities of the detection phase, primarily averages of radar cross sections, the second subsection derives parameters which are useful in the tracking problem. We are concerned with the erratic meander of the target centroid in the radar's field of view (say in azimuth and elevation).

Logically a third and fourth subsection should follow the second one and produce the corresponding parameters in range and range rate (doppler). But these problems were not part of our present assignment. They will be analyzed in future projects. (cf. remarks of Section VIII).

7.1 Simple Statistics

7.1.1 Radar Cross Sections of Fading Targets

Mean Cross Section

$$\bar{\sigma}(\Theta) = \sum_i \sigma_i(\Theta) \quad (3a)$$

In the bistatic case, the quantity Θ is replaced by two angles Θ_t and Θ_r , as in equation (3b) and all following formulae.

$$\bar{\sigma}(\Theta_t, \Theta_r) = \sum_i \sigma_i(\Theta_t, \Theta_r) \quad (3b)$$

Cross Section Peak

$$\sigma_{\text{MAX}} = \left(\sum_i \sqrt{\sigma_i} \right)^2 \quad (4)$$

RMS Cross Section Spread About Mean

$$\sigma_{\text{sd}} = \left(\sum_{i \neq j} \sigma_i \sigma_j \right)^{1/2} \quad (5)$$

7.1.2 Radar Target Glint

Mean Centroid, Using Slow AGC (Effective Target Center)

$$\bar{X} = \frac{\sum_i X_i \sigma_i}{\bar{\sigma}} \quad (6)$$

Corresponding equations hold for the components of X.

$$\bar{x} = \frac{\sum_i x_i \sigma_i}{\bar{\sigma}} \quad (6a)$$

$$\bar{y} = \frac{\sum_i y_i \sigma_i}{\bar{\sigma}} \quad (6b)$$

By definition and equation (1c),

$$\bar{\epsilon} = \frac{\sum_i \left[(x_i - \bar{x}) \sin \theta_r + (y_i - \bar{y}) \cos \theta_r \right] \sigma_i}{r \bar{\sigma}} = 0. \quad (7)$$

Covariance Matrix of Apparent Centroid Motion

$$\overline{(\Delta X \Delta X')} = \frac{1}{4\sigma^2} \left[\sum_{i \neq j} \sigma_i \sigma_j (X_i + X_j - 2\bar{X})(X_i + X_j - 2\bar{X})' \right] \quad (8)$$

In particular,

Standard Deviation of Apparent Centroid in x-Direction

$$x_{sd} = \frac{1}{2\sigma} \left[\sum_{i \neq j} \sigma_i \sigma_j (x_i + x_j - 2\bar{x})^2 \right]^{1/2} \quad (8a)$$

Standard Deviation of Apparent Centroid in y-Direction

$$y_{sd} = \frac{1}{2\sigma} \left[\sum_{i \neq j} \sigma_i \sigma_j (y_i + y_j - 2\bar{y})^2 \right]^{1/2} \quad (8b)$$

Standard Deviation of Measured Error Angle (Slow AGC)

$$\begin{aligned} \epsilon_{sd} = \frac{1}{2r\sigma} \left\{ \sum_{i \neq j} \sigma_i \sigma_j \left[(x_i + x_j - 2\bar{x}) \sin \theta_r + \right. \right. \\ \left. \left. + (y_i + y_j - 2\bar{y}) \cos \theta_r \right]^2 \right\}^{1/2} \quad (9) \end{aligned}$$

7.2 Refined Statistics (Density of Nulls)

7.2.1 Radar Cross Sections of Fading Targets

Mean Number of Lobes per Unit of Parameter

$$\bar{N}_q \{\Delta\sigma; q, p\} = \frac{G_q}{\sigma_{sd}} \left[\sum_{i \neq j} \sigma_i \sigma_j (q F_{ij})^2 \right]^{1/2} \quad (10)$$

In application to specific parameters and cases, Table 1 serves for the computation of G_q , Table 2 for the computation of the term $q F_{ij}$.

Table 1

VALUES OF G_q

Parameter q	Units	G_q
Aspect angle θ	rad	$\frac{k}{2\pi} = \frac{1}{\lambda}$
	deg	$\frac{k}{360}$
Radio frequency f	Hz	$\frac{1}{c}$
	MHz	$\frac{10^6}{c}$
Time t	sec	$\frac{k}{2\pi} \dot{\theta}$

where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{\omega}{c} \text{ is the wave number,}$$

c = speed of light,

$\dot{\theta}$ = rate of change of aspect angle in radians/sec.

The units of λ and c must be compatible with those of σ_1 , σ and $\bar{\sigma}$.

Table 2

VALUES OF $q F_{ij}$

Parameter q	Monostatic Case	Bistatic Case
θ or t	$2[(x_i - x_j) \sin \theta + (y_i - y_j) \cos \theta]$	$[(x_i - x_j) (\sin \theta_t + \sin \theta_r) + (y_i - y_j) (\cos \theta_t + \cos \theta_r)]$
f	$2[(x_i - x_j) \cos \theta - (y_i - y_j) \sin \theta]$	$[(x_i - x_j) (\cos \theta_t + \cos \theta_r) - (y_i - y_j) (\sin \theta_t + \sin \theta_r)]$

One example follows:

Mean Number of Lobes per Degree at Fixed Frequency in the Bistatic Case

$$\bar{N}_{\theta} \{ \Delta \sigma; \theta_t, \theta_r \} = \frac{k}{360 \sigma_{sd}} \left\{ \sum_{i \neq j} \sigma_i \sigma_j \left[(x_i - x_j) (\sin \theta_t + \sin \theta_r) + (y_i - y_j) (\cos \theta_t + \cos \theta_r) \right]^2 \right\}^{1/2} \quad (10a)$$

7.2.2 Radar-Target Glint

Mean Number of Centroid Excursions per Unit of Parameter in the x-Direction

$$\bar{N}_q \{ \Delta x; q, p \} = \frac{G_q}{2 \sigma_{x_{sd}}} \left[\sum_{i \neq j} \sigma_i \sigma_j (x_i + x_j - 2\bar{x})^2 (q F_{ij})^2 \right]^{1/2} \quad (11a)$$

In the y-Direction

$$\bar{N}_q \{\Delta q; q, p\} = \frac{G_q}{2\bar{\sigma}_y \epsilon_{sd}} \left[\sum_{i \neq j} \sigma_i \sigma_j (y_i + y_j - 2\bar{y})^2 (q_{F_{ij}})^2 \right]^{1/2} \quad (11b)$$

While the last two quantities are not directly observable, the following quantity can be measured from the receiver:

Mean Number of Error Fluctuations per Unit of Parameter

$$\bar{N}_q \{\epsilon; q, p\} = \frac{G_q}{2\bar{\sigma}_r \epsilon_{sd}} \left\{ \sum_{i \neq j} \sigma_i \sigma_j \left[(x_i + x_j - 2\bar{x}) \sin \theta_r + (y_i + y_j - 2\bar{y}) \cos \theta_r \right]^2 (q_{F_{ij}})^2 \right\}^{1/2} \quad (12)$$

One example follows:

Mean Number of Error Fluctuations per MHz at
Fixed Angle in the Monostatic Case

$$\bar{N}_r \{\epsilon; \theta\} = \frac{10^6}{c\bar{\sigma}_r \epsilon_{sd}} \left\{ \sum_{i \neq j} \sigma_i \sigma_j \left[(x_i + x_j - 2\bar{x}) \sin \theta + (y_i + y_j - 2\bar{y}) \cos \theta \right]^2 + [(x_i - x_j) \cos \theta - (y_i - y_j) \sin \theta]^2 \right\}^{1/2} \quad (12a)$$

VIII. APPLICATION OF ANALYSIS, CONCLUSIONS AND RECOMMENDATIONS

Application of Analysis

The Conductron Corporation of Ann Arbor, Michigan, has applied the mathematical results of Section VII to a series of specified missiles.* A set of numerical calculations was performed for the output quantities listed below as functions of various RF-wavelengths, aspect angles and polarizations:

$\bar{\sigma}$	eq. (3a)
σ_{MAX}	eq. (4)
σ_{sd}	eq. (5)
\bar{x}	eq. (6a)
x_{sd}	eq. (8a)
$\bar{N}_q\{\Delta\sigma; q, p\}$	eq. (10)
(1.) for $q = 0, p = 0$	
(2.) for $q = f, p = 0$	
$\bar{N}_q\{\Delta x; q, p\}$	eq. (11a)
(1.) for $q = 0, p = 0$	
(2.) for $q = f, p = 0$	

At the same time, the Applied Physics Laboratory under the sponsorship of ARPA carried out a parallel RCS measurement program at Holloman Air Force Base by static ground tests on models of the same specified missiles. Again various RF-wavelengths, aspect angles and polarizations were utilized. What conclusions can be reached from a comparison of the theoretical and experimental results?

*Reference [19], (c) and (d).

Conclusions

Before answering the last question, a few general remarks are in order. We deal with very complex and extremely phase-sensitive structures in the microwave regime. At first sight, some of the output quantities appear to fluctuate wildly and randomly as functions of certain parameters or in response to slight changes of the configuration. It is difficult to assign a well-shaped spectrum to the output signals. A flat spectrum seems to be the closest choice for them at best. This characteristic feature is confirmed by the analysis (see Appendix A.III). It is even more difficult to read, extrapolate or guess an "instantaneous lobing frequency"* from the inspection of the test records. It was indeed this difficulty which led to the present investigation. In the light of these remarks, the following conclusions can be stated:

- (a) There exists a general qualitative agreement between the computed and measured quantities.
- (b) In particular theoretical and experimental results show the same trends. For instance, the lobing frequency is lowest "head-on" i.e., in the direction where the aspect angle θ is zero, it rises with increasing aspect angle and reaches one or several maxima in oblique directions. The maxima are not fixed for the same missile if the radio frequency or polarization are varied. As expected the lobing frequency N_0 increases with higher radar frequencies f , while the other parameters are held fixed.

*The discussion of the class of deterministic and random functions which allow an "instantaneous frequency" will be taken up in a future report.

- (c) Provided a diligent mathematical and numerical analysis is performed, it appears that the accuracy of the important output quantities is sufficient for a statistical radar-design and performance study concerned with detection, track and identification of scintillating targets.

Recommendations

The computed quantities referring to the meander of the centroid, such as \bar{x} (eq. 6a), x_{sd} (eq. 8a), $\bar{N}_q\{\Delta x; q, p\}$ (eq. 11a) et al., have not been verified as yet by experimental measurements. This should be done, preferably in dynamic flight tests.

As mentioned above this analysis should be extended to include the scintillation of the outputs of the range and range rate (doppler) circuits.

The investigation should be generalized to incorporate targets containing non-random features, such as periodic sections, uniformly distributed scatterer segments, etc. (cf. remarks at the end of Sec. A.III).

The effects of body vibrations and of the slower oscillations of the stability, control and guidance loops on the radar scintillation should be examined (cf. discussion in Sec. A.V.1).

Finally, the usefulness of special devices (RF frequency modulators, adaptive narrow-band filters with variable center frequencies, etc.) should be studied. The objective would be to recognize the size, overall shape and attitude of unknown objects or to identify the signature of well known targets. Some of these problems are being studied at the present time and will be documented in the near future.

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APPENDICES

A.I. PRINCIPLES USED IN THE DERIVATION OF THE ANALYTICAL RESULTS OF SECTION 7.1.1

The mathematical results of Section 7.1 entitled "Simple Statistics," Subsection 7.1.1 "Radar Cross Sections of Fading Targets" were derived by the research team of the Conduction Corporation. For details of this analysis, Refs.[5] and [19] should be consulted. Here we shall discuss the basic principles utilized in this work.

- (a) Far-Field Scattering,
- (b) Born Approximation,
- (c) Method of Relative, Random Phase.

Principle (a) is based on our assumptions (5) and (3) of Section VI. It simplifies the problems of wave physics and in many cases allows the use of plane wave fronts.

Principle (b) is well known from scattering problems in quantum mechanics. There it becomes applicable, if the interaction coefficients are so small that double and multiple scattering involving squares and higher powers of the interaction constants can be ignored. In our case multiple scattering can be omitted since we deal with bodies of simple shapes (assumptions (1) and (4) of Section VI).

Principle (c) or the method of relative and random phase will be described next. Following the analysis of Ref. [5], page 25 a.f. and using the notation of Section V, one obtains for the instantaneous RCS of the target:

$$\sigma_p = \left| \sum_{j=1}^N (\sigma_j)^{1/2} \exp(i\phi_j) \right|^2 \quad (\text{A.I.1})$$

where in the monostatic case

$$\Phi_j = -2k[(x_j - \bar{x})\cos \theta - (y_j - \bar{y})\sin \theta] + \bar{\Phi} \quad (\text{A.I.2a})$$

in the bistatic case

$$\Phi_j = -k[(x_j - \bar{x})(\cos \theta_t + \cos \theta_r) - (y_j - \bar{y})(\sin \theta_t + \sin \theta_r)] + \bar{\Phi} \quad (\text{A.I.2b})$$

k is the wave number explained at the bottom of Table 1.

$\bar{\Phi}$ is an absolute, instantaneous phase of the mean centroid.

The most difficult part of the analysis is the job of finding the magnitude of the individual scattering cross sections σ_i . The derivation of σ_p and σ proceeds in three steps. To describe the method, we follow Ref. [5] closely and verbatim in many instances:

First Step - Modeling

We consider the body under investigation as an assembly of many components each of which can be geometrically approximated by a "simple shape" in such a way that the RCS of the simple shape approximates the RCS of the component it models.

Second Step - Computation of Component RCS

It is assumed that the RCS of the "simple shape" can be computed. The rigorous method would require a solution of Maxwell's equations with known but complicated boundary conditions. If an exact solution is not known, one must find an approximate solution by some special method.

Third Step - Combination of Individual Component RCS's

The final step involves the proper combination of the "component cross sections" to yield the estimate of the cross section of the entire body. Here again it is often necessary to find a simplifying mathematical scheme such as the method of "relative random phase." It must be emphasized that mathematical approximations must be justified on physical grounds.

Rewriting eq. (A.I.1) one gets in the monostatic case:

$$\sigma_p = \sum_{i=1}^N \sum_{j=1}^N (\sigma_i \sigma_j)^{1/2} \cos 2k[(x_j - x_i) \cos \theta - (y_j - y_i) \sin \theta]$$

In the notation of Section IV:

$$\sigma_p = \sum_i \sigma_i + 2 \sum_{i < j} (\sigma_i \sigma_j)^{1/2} \cos 2k[(x_j - x_i) \cos \theta - (y_j - y_i) \sin \theta] \quad (\text{A.I.3})$$

It is seen that the absolute, instantaneous phase Φ of the centroid disappears from the equation and therefore becomes irrelevant. This fact justifies the term "relative phase." A particular procedure of combining individual component cross sections is called the "method of random phase." It is based on the assumption (see p. 26 of Ref. [5]) that the many different Φ_j are randomly distributed; then upon averaging over Φ_j one obtains the expression

$$\bar{\sigma} = \sum_i \sigma_i \quad (\text{A.I.4})$$

which is identical with eq. (3a) of Section VII.

The reasoning which led to eq. (A.I.4) appears somewhat vague. However, one has the feeling that the result is physically correct. One might try to strengthen the mathematical argument by

- (a) introducing statistical distributions for the x_j (say Poisson distributions) and independent but identical distributions for the σ_1 (say Rayleigh distributions),
- (b) going to the limit $\frac{L/N}{\lambda} \rightarrow \infty$, (L is the length of the target)
- (c) replacing sums by integrals and averaging.

Then one winds up with expressions containing Dirac's delta functions, giving rise to analytical difficulties.

An alternate method of combining wildly fluctuating signal components will be discussed in Section A.II.

A.II. PRINCIPLES AND METHODS USED IN THE ANALYSIS OF SECTION 7.2

Referring back to the list of principles enumerated in Section VI, we shall make use of the following items:

- (d) Central Limit Theorem,
- (e) Principle of Local Stationarity,
- (f) Lord Kelvin's Principle of Stationary Phase and its Extension.

Item (d) is based on assumption (2) of Section VI. If we deal with many independent scatterers, whose individual RCS's follow independent but equal distributions, and if some other conditions are met (such as the existence of first and second moments), the RCS of the combined signal in the limit $N \rightarrow \infty$ will approximate a Gaussian distribution.* This property is important because it enables us to make use of some results obtained by S. O. Rice in the field of the statistics of the count of zero crossings.

Item (e) is a new concept which was introduced here especially for the purpose of coping with the type of signals encountered in radar target echoes. These signals, in the long run, or averaged over a long interval of time, display a broad, flat spectrum. Yet we know these signals bear an instantaneous statistical character which changes slowly with the parameter (aspect angle θ , time t or frequency f).

The concept of "local stationarity" shall be explained on hand of the signal σ_p (cf. eq. A.I.1).

$$\sigma_p' = \left| \sum_{j=1}^N (\sigma_j)^{1/2} \exp(i\Phi_j) \right|^2 \quad (\text{A.II.1})$$

*Reference [14], p. 215.

For simplicity we choose the monostatic case and a one-dimensional distribution of scatterers. Accordingly:

$$\phi_j = -2k[(x_j - \bar{x}) \cos \Theta] + \bar{\phi} \quad (\text{A.II.2})$$

Then (A.II.1) can be rewritten as follows:

$$\sigma_p(\Theta) = \sum_{i \neq j} (\sigma_i \sigma_j)^{1/2} \cos [2k(x_j - x_i) \cos \Theta] \quad (\text{A.II.3})$$

Θ is the independent parameter of the real signal process. Now we introduce two additional independent virtual variables τ and δ :

$$\sigma_p(\Theta, \tau, \delta) = \sum_{i \neq j} (\sigma_i \sigma_j)^{1/2} \cos [2k(x_j - x_i) \cos(\Theta + \tau + \delta)] \quad (\text{A.II.4})$$

τ is the parameter-shift variable in a statistical ensemble with the range $0 \leq \tau \leq 2\pi$ and the probability density:

$$p(\tau) = \frac{1}{2\pi} \quad (\text{A.II.5})$$

δ is the parameter in a locally stationary ensemble with the range $-\Delta \leq \delta \leq +\Delta$ and the probability density

$$p(\delta) = \frac{1}{2\Delta} \quad (\text{A.II.6})$$

Irrespective of how small we choose Δ , we can always find a large enough k or $k \cdot (x_j - x_i)$, so that the argument or phase ϕ_{ij} of every cos-term describes the full circle from 0 to 2π , even several times. In the limit, if $k(x_j - x_i) \rightarrow \infty$, the process $\sigma_p(\delta)$ appears as a full random process covering infinite phase shift as δ goes through its range $-\Delta \leq \delta \leq +\Delta$. In the following, we shall always assume that we are close to this idealized situation. For lack of other suitable terms, we shall call a process of this type a "peculiar process," i.e., peculiar to certain types of radar echoes.

Indeed, the peculiar processes share with the flat, white noise processes the broad spectrum. However, they appear to be more closely related to, but not identical with the noise processes of Reference [16], eq. (12).

The peculiar processes are not stationary over the full ranges of their parameters (Θ , τ). However, for fixed Θ and τ , we may define a "local stationarity," by taking a "microscopic look" so to speak, at the small range $-\Delta \leq \delta \leq +\Delta$ of the local ensemble. By making a transformation of the form

$$\delta = \frac{\Theta}{\pi/\Delta}$$

we obtain a process of the parameter Θ which is stationary in the range

$$-\pi \leq \Theta \leq +\pi.$$

This procedure is similar to Newton's idea of forming derivatives. To any point P_1 of a curvilinear function choose an infinitesimally near point P_2 on the curve. Connect P_1 with P_2 by a straight line. The infinitesimal segment P_1P_2 can then be blown up by extending it to a tangential line. It is well known that derivatives do not exist for certain classes of real, even continuous functions. In the same manner, stochastic processes do not necessarily possess local stationarity, but the peculiar processes enjoy this property.

The virtual variable τ is used in connection with ordinary stationary, statistical quantities, such as the autocorrelation function, spectrum, etc. The virtual variable δ is used in connection with locally stationary properties, such as "instantaneous frequency," "average number of zero crossings," etc.

We can speak of several types of statistical averages for a quantity z of such a process:

The parameter (time) average is indicated by \bar{z} . The ensemble average is denoted by $\langle z \rangle$. It is well known that for stationary, ergodic processes one has:

$$\bar{z} = \langle z \rangle \quad (\text{A.II.7})$$

We have many real component scatterers. Therefore we could also speak of an assembly average \tilde{z} . In our work assembly averages are not required, but total sums over the assembly are, which in the simplest cases become $N\tilde{z}$. For instance one gets:

$$\sqrt{\sigma_p} = N |\sigma_j|^{1/2} \exp(i\phi_j)$$

The local ensemble average $\{z(\Theta, \tau)\}$ is defined by

$$\{z(\Theta, \tau)\} = \lim_{\Delta \rightarrow 0} \frac{1}{2\Delta} \int_{-\Delta}^{+\Delta} d\delta \, z(\Theta + \tau + \delta) \quad (\text{A.II.8})$$

It is important to note that for our type of processes the assembly sums change with finite shifts of the real parameter Θ and of the ensemble parameter τ . A rigorous mathematical analysis would have to work around Dirac's δ -functions, since the processes considered have a flat spectrum in the real and ensemble domain. Our present objective is limited to reducing expressions which require exorbitant calculational work, to a tractable computational form. Therefore we shall omit further mathematical and theoretical exercises at this time.

Generalization of the Principle of Stationary Phase

The principle of stationary phase was originally enunciated by Lord Kelvin as a mathematical artifice to approximate infinite integrals of wildly fluctuating functions of the type occurring in the hydrodynamic theory of impulsive disturbances or wave groups of small spatial extension. These waves possess a broad spectrum. This principle was later cast in a mathematical rigorous form (see Ref. [15]).

We shall state this principle in 2 slightly more general forms applicable to our locally stationary processes.

First Version of Kelvin's Principle

Given a finite function $f\{\Phi\}$ of bounded variation, such as the cos-function. The argument is called phase. The latter itself is a function of two sets of variables: $\{m\}$ and $\{x, y, \dots\}$

$$\Phi = \Phi(m; x, y, \dots)$$

m is continuous and x, y, \dots assume discrete values only.

We form infinite integrals over the first variable and finite, but large summations over the second set.

$$E = \int_{-\infty}^{\infty} dm \sum_{i,j=1}^N f\{\Phi(m; x_i y_j, \dots)\} \quad (\text{A.II.9})$$

If the phase Φ fluctuates wildly as a function of m , then one can obtain a good approximation of E

(a) either by integrating over a short range of m :

$$m_0 - \Delta \leq m \leq m_0 + \Delta \quad \text{and, or}$$

(b) by summing over a smaller subset $x_{1_0}, x_{2_0}, \dots, y_{1_0}, y_{2_0}, \dots$ of the second set, so long as the stationarity of phase holds for the two subsets (a) and (b), i.e., so long as the following equation is valid:

$$\frac{\partial}{\partial m} \{m_0; x_{i_0}, y_{j_0}, \dots\} = 0 \quad \text{for } m_0 \quad \text{and all } x_{i_0}, y_{j_0}. \quad (\text{A.II.10})$$

Then

$$E \sim \int_{m_0 - \Delta}^{m_0 + \Delta} dm \sum_{i=i_0} \sum_{j=j_0} f\{\Phi(m; x_i, y_j, \dots)\} \quad (\text{A.II.11})$$

The second part (b) of the proposition constitutes the generalization of Kelvin's principle.

Example: Let us apply principle (f) to our locally stationary signal σ_p (eq. A.II.4). The dummy variable δ is substituted for m and x_i, x_j for x_1, y_j . The variables Θ and τ are not summed here. In our case δ runs through a very short range already. Therefore part (b) is expected to play a major role in the simplification process.

Equation (A.II.10) becomes

$$0 = \frac{d}{d\delta} [2k(x_{i_0} - x_{j_0}) \cos(\Theta + \tau + \delta_0)] \quad \text{or} \\ (x_{i_0} - x_{j_0}) \sin(\Theta + \tau + \delta_0) = 0 \quad (\text{A.II.12})$$

Since Θ and τ are arbitrary this equation can only be satisfied if $i_0 = j_0$ or starting with the original set $\{i, j\}$ one forms the subset $\{i_0, j_0\}$:

$$i_0 = i = 1, 2, \dots, N \\ j_0 = i$$

The local ensemble average of σ_p becomes

$$\{\sigma_p\} = \sum_{i=1}^N \sigma_i \quad (\text{A.II.13})$$

Since the parameter Θ disappeared from the right hand side of eq. (A.II.13), this expression is also the parameter-average $\bar{\sigma}$:

$$\bar{\sigma} = \sum_{i=1}^N \sigma_i \quad (\text{A.II.14})$$

This is the result of Ref. [5] and the content of eq. (A.I.4). In later examples we shall form local ensemble averages that are not identical with the parameter averages. The use of the extension of Kelvin's principle somewhat strengthens the intuitive argument given on pg. 26 of Ref. [5]. The validity of Kelvin's principle was proved for a large class of oscillating functions by G. N. Watson.* The extension of this principle should be treated in a more rigorous form too. But this job exceeds the scope of our present assignment.

Second Version of Kelvin's Principle

We shall now describe a second version of Kelvin's principle of stationary phase that will be utilized in the following section. In the previous case the phase function contained 2 sets of variables: a continuous variable and a set of discrete variables. Accordingly the total summation consisted of an integration and a finite summation over discrete variables. In the sequel the phase function contains two sets of discrete variables and the total summation consists of two finite summations over discrete variables. In particular each summation could be a double sum over discrete variables, as follows:

$$E = \sum_{i,j=1}^N \sum_{n,m=1}^N f\{\Phi(x_i, x_j; x_n, x_m)\} \quad (\text{A.II.15})$$

Again, if the phase Φ fluctuates wildly as a function of the variables, one may get a fair approximation of E , if one sums only over a subset $\{x_{i_0}, x_{j_0}; x_{n_0}, x_{m_0}\}$ of the set $\{x_i, x_j; x_n, x_m\}$ for which the phase is stationary or constant:

$$\Phi(x_{i_0}, x_{j_0}; x_{n_0}, x_{m_0}) = \text{constant for any choice of} \quad (\text{A.II.16})$$

the x -variables within the subset.

It turns out that in the special applications of the following section, the constant vanishes due to the fact that the component scatterers x_i are assumed to be distributed in a random fashion, i.e., numerically placed at incommensurate distances.

*Reference [17]

A.III. APPLICATION OF RICE'S THEORY TO THE COMPUTATION OF RADAR-CROSS SECTION NULLS

The radar cross section σ_p of a target* which satisfies the conditions of Section VI, can be computed by the method of relative phase and is given by eq. (A.I.3). With slightly modified notation it becomes:

$$\sigma_p = \sum_1 \sigma_1 + \sum_{i \neq j} (\sigma_1 \sigma_j)^{1/2} \cos 2k [(x_j - x_i) \cos \theta - (y_j - y_i) \sin \theta] \quad (\text{A.I.3})$$

$\sigma_p = \sigma_p(\theta)$ can be considered as a stochastic parameter process with the aspect angle θ playing the role of the independent parameter. According to eq. (A.I.4), the mean $\bar{\sigma}$ of σ_p is given by

$$\bar{\sigma} = \sum_1 \sigma_1 \quad (\text{A.I.4})$$

Therefore

$$\Delta\sigma = \sigma_p - \bar{\sigma} = \sum_{i \neq j} (\sigma_1 \sigma_j)^{1/2} \cos 2k [(x_j - x_i) \cos \theta - (y_j - y_i) \sin \theta] \quad (\text{A.III.1})$$

is an unbiased parameter process. Moreover, it approximates an unbiased Gaussian process for the following reasons: If we hold i fixed, the r.h.s. of eq. (A.III.1) is a sum of N terms T_{ij} for $j = 1, 2, \dots, N$. Each of these terms follows an independent distribution (according to assumption 2 of Section VI). Furthermore all finite moments of each term exist, since we deal with a finite number of scatterers and the total extension of the target is bounded. Therefore the conditions of the central limit theorem are met (see Ref. [14], pg. 215). Hence each term T_i with

$$T_i = \sum_{j=1}^N T_{ij}$$

approximates a Gaussian distribution as N grows toward infinity.

*Again the monostatic case is analyzed here for simplicity.

The parameter process $\Delta\sigma$ can be written as

$$\Delta\sigma = \sum_{i=1}^N T_i$$

i.e., as a sum of N Gaussian processes. Therefore $\Delta\sigma$ itself -- according to a well known theorem of statistics -- will be Gaussian or -- more precisely -- will approximate a Gaussian process, as N tends toward infinity. $\Delta\sigma$ is not a stationary process, that is, its statistical properties are not invariant toward a finite shift of the parameter θ . However, we learned in Section A.II that it could be considered as a locally stationary process. For any fixed choice of θ we can form a local ensemble with the parameter δ and range $-\Delta \leq \delta \leq +\Delta$ for which $\Delta\sigma$ behaves like a stationary process.

The process $\Delta\sigma(\theta)$ defined by eq. (A.III.1) and described in the previous discussion shall be called an "unbiased peculiar process." To bring out the essential features of the analysis and save space at the same time, we shall assume that all the component scatterers are distributed over a linear segment. The equations of Section VII (Summary of Mathematical Results), however, cover the general two-dimensional case.

The Auto-Correlation Function of an Unbiased Peculiar Process

The second version of Lord Kelvin's principle of stationary phase (see Section A.II) shall now be applied to the computation of the auto-correlation function $\rho(\tau, \theta)$ of the process $\Delta\sigma(\theta)$, c.f. eq. (A.III.1). Using the average over a local ensemble, one gets:

$$\rho_{\Delta\sigma}(\tau, \theta) = \frac{1}{2\Delta} \int_{-\Delta}^{+\Delta} d\delta \Delta\sigma(\theta+\delta) \Delta\sigma(\theta + \tau + \delta) \quad (\text{A.III.2})$$

Substituting eq. (A.III.1) in (A.III.2) and reducing the problem to one dimension, one has:

$$\rho_{\Delta\sigma}(\tau, \Theta) = \frac{1}{2\Delta} \int_{-\Delta}^{+\Delta} d\delta \ 4 \sum_{i < j} \sum_{n < m} (\sigma_i \sigma_j)^{1/2} (\sigma_n \sigma_m)^{1/2} \\ \times \cos[2k \cos(\Theta + \delta)(x_j - x_i)] \cos[2k \cos(\Theta + \delta + \tau)(x_m - x_n)]$$

or

$$\rho_{\Delta\sigma}(\tau, \Theta) = \frac{1}{2\Delta} \int_{-\Delta}^{+\Delta} d\delta \ 2 \sum_{i < j} \sum_{n < m} (\sigma_i \sigma_j)^{1/2} (\sigma_n \sigma_m)^{1/2} \\ \times \{ \cos[2k \cos(\Theta + \delta)(x_j - x_i)] + 2k \cos(\Theta + \delta + \tau)(x_m - x_n) \} \\ + \cos[2k \cos(\Theta + \delta)(x_j - x_i) - 2k \cos(\Theta + \delta + \tau)(x_m - x_n)] \}$$

(A.III.3)

The radar cross sections σ_i of the component scatterers satisfy a certain distribution. It is well known that the strong scintillation of radar echoes is caused primarily by the interference of phases between the component scatterers rather than by the fluctuation of the σ_i of the individual scatterers.* Be it as it may, let us assume temporarily that the σ_i are constant and equal and their variations are absorbed by slight changes of the large kx_i quantities.

Since $j > i$ and $m > n$, the coefficients of the cos terms in the square bracket of the second line of eq. (A.III.3) are positive and therefore the whole expression in the second line will vanish since the heavily fluctuating terms cancel one another. A similar situation prevails for the expression of the

*This property is discussed in Ref. [1], pg. 202. The fact that the individual σ_i do not depend on Θ is explained by the large width of the scattering lobe patterns of the individual scatterers, in contrast to the narrow lobes produced by an array of length L . The constancy of the individual σ_i as a function of the radio frequency is pointed out in Ref. [22], pg. 573.

third line so long as $\tau > 0$, since we assumed a completely random and incommensurate distribution of the x_i . However, if $\tau = 0$, the situation is different. Then there exists a subset $\{x_{10}, x_{j0}; x_{n0}, x_{m0}\}$ of the set $\{x_1, x_j; x_n, x_m\}$ for which the phase (term in square bracket) is constant, even zero for all combinations. The subset is:

$$\left. \begin{array}{l} x_{10} = x_1 \\ x_{j0} = x_j \\ x_{n0} = x_{10} \\ x_{m0} = x_{j0} \end{array} \right| \quad i, j = 1, 2, \dots, N \quad (\text{A.III.4})$$

The auto-correlation function becomes:

$$\rho_{\Delta\sigma}(\tau, \Theta) = \begin{cases} \sum_{i \neq j} \sigma_i \sigma_j & \text{for } \tau = 0 \\ 0 & \text{for } \tau \neq 0 \end{cases} \quad (\text{A.III.5})$$

As N tends toward infinity, $\rho_{\Delta\sigma}(\tau, \Theta)$ approximates a Dirac delta-function for all values of Θ . Therefore the spectrum becomes wide-banded and flat. This property can also be proved by other methods* and certainly was expected. However, it is highly surprising that one can define an instantaneous frequency or rate of lobing for the "peculiar process." This concept is based on the noise theory of S. O. Rice (Ref. [10]).

The Average Rate of Nulls

For a Gaussian, stationary unbiased process $z(t)$ with correlation function $\rho_z(\tau)$, Rice derived the quantity $\bar{N}_t\{z\}$, i.e., the average number of zero crossings with positive slope (number of lobes) per unit time:

* Compare discussion in Appendix A.V.2.

$$\bar{N}_t(z) = \frac{1}{2\pi} \left\{ \frac{-\ddot{\rho}_z(\tau)}{\rho_z(\tau)} \right\}_{\tau=0}^{1/2} \quad (\text{A.III.6})$$

This concept can be carried over directly to Gaussian, locally stationary, parameter processes. For the unbiased peculiar processes one obtains:

$$\bar{N}_\theta(\Delta z; \theta) = \frac{1}{2\pi} \left\{ \frac{-\ddot{\rho}_{\Delta z}(\tau, \theta)}{\rho_{\Delta z}(\tau, \theta)} \right\}_{\tau=0}^{1/2} \quad (\text{A.III.6a})$$

Note: In the non-stationary, though "locally stationary" processes all quantities depend on the instantaneous value of the parameter θ . The surprising property of the peculiar processes mentioned above can now be explained by the fact that while $\rho(\tau \rightarrow 0)$ and $\dot{\rho}(\tau \rightarrow 0)$ tend toward infinity, their ratio remains finite and gives the desired value.

We start with eq. (A.III.3) and simplify the expression on the r.h.s. somewhat by omitting the second line which oscillates heavily and averages to zero and skipping the integral over the range of the local ensemble variable δ which for the specific conditions of our example cancels out.

$$\rho_{\Delta\sigma}(\tau, \theta) = 2 \sum_{i < j} \sum_{n < m} (\sigma_i \sigma_j)^{1/2} (\sigma_n \sigma_m)^{1/2} \times \cos[2k \cos \theta (x_j - x_i) - 2k \cos(\theta + \tau) (x_m - x_n)] \quad (\text{A.III.3a})$$

Differentiating eq. (A.III.3a) twice with respect to τ , gives:

$$\begin{aligned} \ddot{\rho}_{\Delta\sigma}(\tau, \theta) = & -2 \sum_{i < j} \sum_{n < m} (\sigma_i \sigma_j)^{1/2} (\sigma_n \sigma_m)^{1/2} \\ & \times \left\{ \cos 2k[\cos \theta (x_j - x_i) - \cos(\theta + \tau) (x_m - x_n)] \cdot [2k \sin(\theta + \tau) (x_m - x_n)]^2 \right. \\ & \left. + \sin 2k[\cos \theta (x_j - x_i) - \cos(\theta + \tau) (x_m - x_n)] \cdot [2k \cos(\theta + \tau) (x_m - x_n)] \right\} \end{aligned} \quad (\text{A.III.7})$$

Now, we keep n and m fixed and vary i and j . With our special conditions (very large k , random distribution of the x_i , relative constancy of σ_i) the summation over i, j of the terms in the second line which fluctuate wildly, average to zero except for a subset (A) $\{i_0, j_0; n_0, m_0\}$ of $\{i, j; n, m\}$:

$$\left. \begin{aligned} x_{n0} &= x_n \\ x_{m0} &= x_m \\ x_{i0} &= x_{n0} \\ x_{j0} &= x_{m0} \end{aligned} \right| \quad (A) \quad n, m = 1, 2, \dots, N \quad (A.III.8)$$

For the subset the phases of the cos terms in the second line disappear, as τ goes to zero. The same reasoning holds for the third line. Therefore the third line vanishes since $\sin \Phi_{n,m} = 0$ if $\Phi_{n,m} = 0$. Combining these results and interchanging i, j with n, m one finally gets:

$$\bar{N}_\Theta(\Delta\sigma; \Theta) = \frac{1}{2\pi} \left\{ \frac{\sum_{i \neq j} (\sqrt{\sigma_i} \sqrt{\sigma_j})^2 [2k \sin \Theta(x_j - x_i)]^2}{\sum_{i \neq j} (\sqrt{\sigma_i} \sqrt{\sigma_j})^2} \right\}^{1/2} \quad (A.III.9)$$

This result can be easily extended to two dimensions by assuming a two-dimensional distribution of the scatterers, random and independent in x and y . Furthermore instead of using the aspect angle Θ as the independent parameter, one may substitute the radio frequency f or the time t or any other suitable variable q . In the auto-correlation function $\rho_{\Delta\sigma}(\tau)$ one has to substitute $\tau = \tau_f$, $\tau = \tau_t$ or $\tau = \tau_q$ for $\tau = \tau_\Theta$. Finally in the derivation of $\bar{N}_q(\Delta\sigma; q)$ one has to apply the second derivative of $\rho(\tau_q)$ with respect to τ_q . The results are summarized in eq. (10) combined with Tables 1 and 2 of Section 7.2. Note: While numerator and denominator of the r.h.s. of eq. A.III.9 tend toward infinity as N grows over all bounds, their ratio remains finite and yields a finite and reasonable value, as was shown by a numerical analysis carried out by the Conductron Corporation.

Effect of Non-Random Distribution of the Scatterers

If the scatterers are not randomly distributed, then the reasoning that led from eq. A.III.3 to eq. A.III.5, breaks down. Supposing the scatterers display some periodic features as follows:

$$\sigma_{i+lp} = \sigma_i$$

$$x_{i+(l+1)p} - x_{i+lp} = x_{i+p} - x_i = \frac{L}{M+1}$$

where p is a fixed integer counting the number of scatterers within a period and l is a running integer $0 \leq l \leq M$ counting the number of periodic sections.* Then one can enlarge the subset (A.III.4) by adding:

$$\left. \begin{array}{l} x_{no} = x_{io} + lp \\ x_{mo} = x_{jo} + lp \end{array} \right| \quad (B) \quad l = 1, 2, \dots, M$$

Furthermore there will exist certain discrete values of τ , say $\tau_1, \tau_2, \dots, \tau_l, \dots$ for which additional subsets $S_1, S_2, \dots, S_l, \dots$ with stationary phase are associated, for instance for τ_l one can construct a subset S_l , as follows:

$$\left. \begin{array}{l} x_{io} = x_i \\ x_{jo} = x_j \\ x_{no} = x_{io} \\ x_{mo} = x_{jo} + lp \end{array} \right| \quad (S_l) \quad i, j = 1, 2, \dots, N \quad (A.III.10) \\ l \text{ is fixed.}$$

* Each index k appearing in the subscripts, should satisfy the condition: $0 \leq k \leq N$. If this condition is not met, replace k by an integer k' which satisfies this inequality and the following relation:

$$k' \equiv k \pmod{N}$$

so that

$$\cos \theta \cdot (x_{j0} - x_{i0}) - \cos (\theta + \tau_l) (x_{m0} - x_{n0}) = 0, \quad (\text{A.III.11})$$

In addition one can also form subsets of type (B).

The correlation function becomes a sum of Dirac delta functions:

$$\rho(\tau) = \sum_{l=0}^M \delta(\tau - \tau_l) a_l \quad (\text{A.III.12})$$

and the radar echo pattern assumes the well known finger-like or fan-like appearance.

A.IV. EFFECT OF AGC ON THE CHARACTER OF CENTROID OSCILLATIONS

1. Instantaneous Centroid for Slow AGC

Let $k = \frac{2\pi}{\lambda}$ be the wave number and let $\Phi_i = k(P_i - P_0)$ for $i = 1, 2, \dots, N$ be the relative phase shift between scatterers i and 0 . $P_0, P_1, \dots, P_1, \dots, P_N$ denote the two-way paths from the radar to the individual scatterers $S_0, S_1, \dots, S_1, \dots, S_N$ and back to the radar. In the monostatic case, the paths run from the tracking radar to the scatterer and back to the same radar. In the bistatic case, the paths extend, e.g., from the illuminator to the scatterer and back to the seeker of a semi-actively homing missile. S_0 is a reference station, say the nose tip of the target or the average centroid position.

In order to determine the position of the centroid, one has to define two vectors V and V_x and one square matrix M .

The matrix M is defined by its general element:

$$M_{ij} = \cos(\Phi_i - \Phi_j); \quad i, j = 1, 2, \dots, N \quad (\text{A.IV.1})$$

where Φ_i is the phase angle defined by eq. (A.I.2a).

The vectors V and V_x are defined by the following columns:

$$V = \begin{bmatrix} \sqrt{\sigma_1} \\ \sqrt{\sigma_2} \\ \vdots \\ \sqrt{\sigma_1} \\ \vdots \\ \sqrt{\sigma_N} \end{bmatrix} \quad V_x = \begin{bmatrix} \sqrt{\sigma_1} x_1 \\ \sqrt{\sigma_2} x_2 \\ \vdots \\ \sqrt{\sigma_1} x_1 \\ \vdots \\ \sqrt{\sigma_N} x_N \end{bmatrix} \quad (\text{A.IV.2})$$

$\sigma_1, \sigma_2, \dots, \sigma_1, \dots, \sigma_N$ are the radar cross sections of the individual component scatterers as defined in Section IV. The quantities x and x_1 are used in the present section as generic terms that might represent either the x -positions, the y -positions or the error angles ϵ of individual scatterers and centroids, as defined in Section IV.

Making use of the assumptions listed in the beginning of Section VI, one may write the position of the instantaneous centroid x_s for slow AGC as follows:

$$x_s = \frac{V' M V_x}{\langle V' M V \rangle} \quad (\text{A.IV.3})$$

The position of the average centroid \bar{x}_s for slow AGC is given by:

2. Average Centroid for Slow AGC

$$\bar{x}_s = \frac{\langle V' M V_x \rangle}{\langle V' M V \rangle} \quad (\text{A.IV.4})$$

The operator $\langle F(\Phi_1, \Phi_2, \dots, \Phi_N) \rangle$ is defined, as follows:

$$\langle F \rangle = \frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d\Phi_1 d\Phi_2 \dots d\Phi_N F(\Phi_1, \Phi_2, \dots, \Phi_N) \quad (\text{A.IV.5})$$

The denominator of eqs. (A.IV.3) and (A.IV.4) is equal to the mean radar cross section $\bar{\sigma}$ which is given by eq. (3a) of Section 7.1:

$$\bar{\sigma} = \sum_1 \sigma_1 \quad (3a)$$

Combining eqs. (A.IV.1) to (A.IV.5) and substituting eq. (3a) gives:

$$\bar{x}_s = \frac{\sum_{i=1}^N x_i \sigma_i}{\bar{\sigma}} \quad (\text{A.IV.6})$$

This formula is identical with eq. (6) of Section 7.1.2. All the results of Section 7.2.2 entitled "Radar-Target Glint" were derived from it by applying conventional, elementary statistics combined with Rice's theory of zero-crossings, as expounded in Appendix A.III.

It should be emphasized here that the mathematical expressions for the effects of AGC, as given by eqs. (A.IV.3) and (A.IV.4), are simplified formulations of the actual AGC response. A more rigorous formulation must account (1) for the fact that the AGC filter has a dynamic response given by an operator or a transfer function, (2) for the real gain control which is not the inverse of the AGC filter output but is represented by a more complicated function.

3. Instantaneous Centroid for Fast AGC

A simplified expression for the position of the instantaneous centroid x_F under the action of fast AGC is given by:

$$x_F = \frac{V' M V x}{V' M V} \quad (\text{A.IV.7})$$

Substituting eqs. (A.IV.1) and (A.IV.2) in (A.IV.7) gives:

$$x_F = \frac{\sum_{i=1}^N \sum_{j=1}^N x_i \sqrt{\sigma_i} \sqrt{\sigma_j} \cos (\phi_i - \phi_j)}{\sum_{i=1}^N \sum_{j=1}^N \sqrt{\sigma_i} \sqrt{\sigma_j} \cos (\phi_i - \phi_j)} \quad (\text{A.IV.8})$$

4. Average Centroid for Fast AGC

The position of the average or effective centroid under the action of fast AGC becomes:

$$\bar{x}_F = \left\langle \frac{V' M V x}{V' M V} \right\rangle \quad (\text{A.IV.9})$$

Combining eqs. (A.IV.5) and (A.IV.8) one obtains:

$$\bar{x}_F = \frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d\phi_1 \dots d\phi_N \frac{\sum_{i=1}^N \sum_{j=1}^N x_i \sqrt{\sigma_i} \sqrt{\sigma_j} \cos(\phi_i - \phi_j)}{\sum_{i=1}^N \sum_{j=1}^N \sqrt{\sigma_i} \sqrt{\sigma_j} \cos(\phi_i - \phi_j)} \quad (\text{A.IV.9a})$$

This multiple integral is difficult to evaluate and only partial results are known to this author.

James Hanson and collaborators of APL investigated problems of this kind. In one study they applied the following restrictions:

- (a) The analysis is linearized. Most angles are very small.
- (b) All the α_i are variable but correlated. They change as linear functions of time and stay in commensurate ratios.

Then one gets the following result:

$$x_1 \leq \bar{x}_F \leq x_N .$$

The present author attacked this problem in a different way. Instead of using vectors and matrices, he cast the expression (A.IV.9) in the form of complex variables. Then the following result could be easily proved:

If there exists a dominant component scatterer S_n with overpowering radar cross section σ_n , i.e., for

$$\sigma_n > \sigma_1 + \sigma_2 + \dots + \sigma_{n-1} + \sigma_{n+1} + \dots + \sigma_N \quad (\text{A.IV.10})$$

one gets

$$\bar{x}_F = x_n . \quad (\text{A.IV.11})$$

This fact is well known for $N = 2$. If a radar applying "fast" or "instantaneous" AGC tracks two unequal targets within the linear regime of its beam width, then the stronger target becomes dominant and pulls the radar axis to its own position. For three or more unequal targets this fact does not hold true, unfortunately. Inequality (A.IV.10) indicates how strong one single target has to be in order to pull the radar axis into its own position.

If one deals with a target that consists of a dominant scatterer and a series of minor echo sources, it often seems desirable to hit the dominant scatterer. If this is the preferred tactic, then the use of fast AGC appears advantageous.

Muchmore (Ref. [1]) investigated the effects of AGC on radar target scintillation. The author assumed the following conditions:

- (a) Simplified target model (one-dimensional, uniform echo density),
- (b) Small error angles,
- (c) No target maneuvers,
- (d) No amplitude noise, no receiver noise, no jamming noise.

Under these conditions Muchmore arrived at the conclusion that slow AGC is superior. He showed that very fast AGC may increase the scintillation noise density by a factor of approximately three.

Of course, if these conditions are relaxed, a different situation will arise. In the example given by the present author above, condition (a) is rescinded. The craft that possesses one-dimensional, uniform echo density, is replaced by a target that contains a dominant scatterer and minor echo sources. In this case, fast AGC is advantageous.

References [7], [23] and [24] investigated the effects of AGC under general conditions where restrictions (a), (b), (c), and (d) are partially or completely relaxed. All three favor

fast AGC. Their conclusion is that under practical tracking conditions a fast-acting AGC will give better tracking performance.

However, if one can afford a sophisticated, flexible system, it might be worth-while to adapt the AGC to the instantaneous situation. For instance, during the terminal flight of a missile which is homing on a long target, glint might well be the dominant noise source. If so, it would pay off to switch the automatic gain control to a setting with a longer AGC time constant.

A.V. ADDITIONAL NOTES CONCERNING OUR ANALYSIS

1. Effects of Vibrations

In the derivation of the electrical phase angles Φ_j of the individual scatterers (see eq. A.I.2a and A.I.2b) we followed the analyses of References [5] and [19]. In the monostatic case one gets:

$$\Phi_j = -2k[(x_j - \bar{x}) \cos \Theta - (y_j - \bar{y}) \sin \Theta] + \Phi \quad (\text{A.I.2a})$$

Some authors (see References [1], [21], and [22]) modify the expressions for the individual phase angles slightly, as follows:

$$\Phi_j = -2k[(x_j - \bar{x}) \cos \Theta - (y_j - \bar{y}) \sin \Theta] + \alpha_j \quad (\text{A.V.1})$$

where the α_j are statistically independent random variables that follow a uniform distribution:

$$p(\alpha_j) = \frac{1}{2\pi} \quad \text{for} \quad 0 \leq \alpha_j \leq 2\pi \quad (\text{A.V.2})$$

In the target model by J. J. Freeman (see Ref. [21]) the random phases α_j are supposed to account for structural vibrations and possibly other random phenomena. The same author made a very diligent study of the effects of these random angles on tracking performance. One of his results is that their effect on a number of mean statistical output quantities (such as the effective target center, correlation functions and spectra) is negligible. This result is gratifying to our studies, inasmuch as we did not include the important effects of structural vibrations in the preceding analysis. However, it can be easily shown that if we had introduced expression (A.V.1) containing the random phases α_j in our analysis, instead of equations (A.I.2a) and (A.I.2b), we would have wound up with the same results. For instance, in

the example of Section A.II, taking the derivative of the phase function and equating it to zero (leading to eq. (A.II.12)) would have eliminated the constant random phases $(\alpha_{ic} - \alpha_{jo})$. In the example of Section A.III, selecting the subset (A.III.4) of terms and adding would have canceled the terms of random phases $[(\alpha_j - \alpha_i) - (\alpha_m - \alpha_n)]$ in eq. (A.III.3).

This fact strengthens our argument for the application of Kelvin's principle and its extensions in our analysis, since we used an approach to the problem which is quite distinct from the method of Reference [21]. Yet we arrived at the same result in the case of the effects of vibrations.

2. Remarks Concerning Spectra

As explained in Section II, spectra are not included in our final results because they are characteristic properties of stationary processes. We used the concepts of Rice's noise theory, such as the average count of zero crossings and other instantaneous statistical quantities which are applicable to locally stationary processes. Since we only needed the ratio of spectra in the intermediate results, we could simplify our analysis by making full use of assumptions (2) and (3) of Section VI. We idealized the radar echo signals by assuming:

$$\frac{L/N}{\lambda} \rightarrow \infty \quad (\text{A.V.3})$$

where L is the length of the target, N is the number of scatterers and λ is the RF wavelength. As a consequence of the limiting relation (A.V.3), the spectra of the radar echo signals became wide-banded and flat, similar to the spectra of "white Gaussian noise processes."

In contrast to our work, References [1], [21] and [22] computed spectra as part of their final results. Muchmore (Ref. [1]) applied the conditions enumerated in Appendix A.IV, pg. 52, and computed for the normalized spectrum of the IF target signal the following expression:

$$W_{IF}(f) = \begin{cases} \frac{1}{2f_m} & \text{for } f_o - f_m \leq f \leq f_o + f_m \\ 0 & \text{otherwise} \end{cases} \quad (A.V.4)$$

The normalized video spectrum becomes:

$$W_V(f) = \begin{cases} \frac{1}{f_m} \left(1 - \frac{f}{2f_m}\right) & \text{for } 0 \leq f \leq 2f_m \\ 0 & \text{otherwise} \end{cases} \quad (A.V.5)$$

The scintillation bandwidth f_m is given by:

$$f_m = \frac{L}{\lambda} \Omega \text{ cps.} \quad (A.V.6)$$

L is the length of a slender, strip-like target which is positioned at right angles to the line-of-sight and turns with a constant rate of Ω radians per sec, f_o is the carrier frequency.

Freeman (Ref. [21]) computed the spectrum of the angular-error signal under similar conditions except that in his case the probability is 1/2 that the target turns with a positive rate ($+\Omega$) and 1/2 that it rotates with a negative rate ($-\Omega$). The spectrum in normalized form* becomes:

$$W_E(f) = \begin{cases} \frac{1}{2f_m} \left[1 + \left(1 - \frac{f}{f_m}\right)^3\right] & \text{for } 0 \leq f \leq 2f_m \\ 0 & \text{otherwise} \end{cases} \quad (A.V.7)$$

Again, the scintillation bandwidth before the filtering action of the servo, is given by (A.V.6). There is no discrepancy between these results and the findings of our Section A.III.

* This result is identical with eq. (21) of Muchmore (Ref. [1]).

If $\frac{L/N}{\lambda}$ tends toward infinity (relationship A.V.3), a fortiori $\frac{L}{\lambda}$ tends toward infinity. Hence the bandwidths f_m of the aforementioned spectra due to eq. (A.V.6) grow over all bounds, in agreement with eq. (A.III.5), computed for $N \rightarrow \infty$.

It is interesting to note that the average time rate of counts \bar{N}_t is simply related to f_m by:

$$\bar{N}_t = c \cdot f_m \quad (\text{A.V.8})$$

where c is a function of the target configuration, of the pertinent variables (θ , f , t) and of the quantity that is counted. If the target resembles the model of references [1], [21] and [22] (one-dimensional, uniform distribution of scatterers), then the constant c can be readily computed. This job is being carried out at the present time and will be documented in a separate report.

3. The Use of a Square-Law Detector

The present analysis (see assumption (8) of Section VI) and most pertinent References ([1], [6], [21], [22]) use square-law detectors in their investigations of radar target scintillation. Since many modern radar receivers apply linear envelope detectors, it is well to dedicate a few thoughts to the implications of this procedure.

The situation can best be summed up by a statement of M.I. Skolnik (Reference [26], pg. 431) and we quote verbatim: "In general, the difference between the two (detectors) is small and the detector law in any analysis is usually chosen for mathematical convenience." This fact is borne out by the mathematical derivation and graphical description of the continuous parts of the spectra of the two detector outputs, in Reference [25]. Comparison of Figures 3.13 and 3.14 of this reference indicates that there is indeed little difference in the shapes of the two spectra.

Similar to our discussion regarding the selection of the AGC (Sec. A.IV), the choice of the type of detector is dictated by the tactical situation and the noise environment. If one can afford a sophisticated, adaptive system, a more detailed analysis is warranted. The following remarks are added with this objective in mind.

Delano (Ref. [6]) made a study for the case where angular scintillation is dominant. He obtained the following result: Let η be the rms centroid fluctuation about the mean radar center for a system using a linear envelope detector and η' the equivalent quantity for a square-law detector. Then

$$\frac{\eta'}{\eta} = \sqrt{\frac{\pi}{4}} = .88. \quad (\text{A.V.9})$$

This improvement of 12% for a square-law detector applies to the final stages of missile flight where target glint might become dominant.

What is the situation at the early stages of flight, say during detection and acquisition? J. I. Marcum (Ref. [3], pg. 211) made a study of the optimum detector law. For the early stages of flight where receiver noise is dominant and the S/N-ratio is small, he found that a square-law detector closely approximates the ideal device. In contrast, for the cases where the S/N-ratio is large (say during later stages of tracking) a linear envelope detector gives ideal performance. This result agrees with the findings of R. A. Smith (Reference [28]). Let ρ_{IF} and ρ_V be the signal-over-noise ratios for the IF-stage and video-stage respectively, using a linear detector, and let ρ'_{IF} and ρ'_V be the equivalent terms for a square-law detector. Then for large signal-over-noise ratios, i.e., $\rho_{IF} = \rho'_{IF} \rightarrow \infty$ he finds:

$$\frac{\rho_V}{\rho_{IF}} = 2 \quad \text{and} \quad \frac{\rho'_V}{\rho'_{IF}} = \frac{1}{2}. \quad (\text{A.V.10})$$

The improvement by a factor of 4 in favor of the linear detector is not as significant as it appears, because for noiseless signals, the type of detector becomes irrelevant.

However, in a practical problem, the specific noise environment should be carefully scrutinized and the system designed accordingly.

4. Remarks Concerning the Counting of Lobes

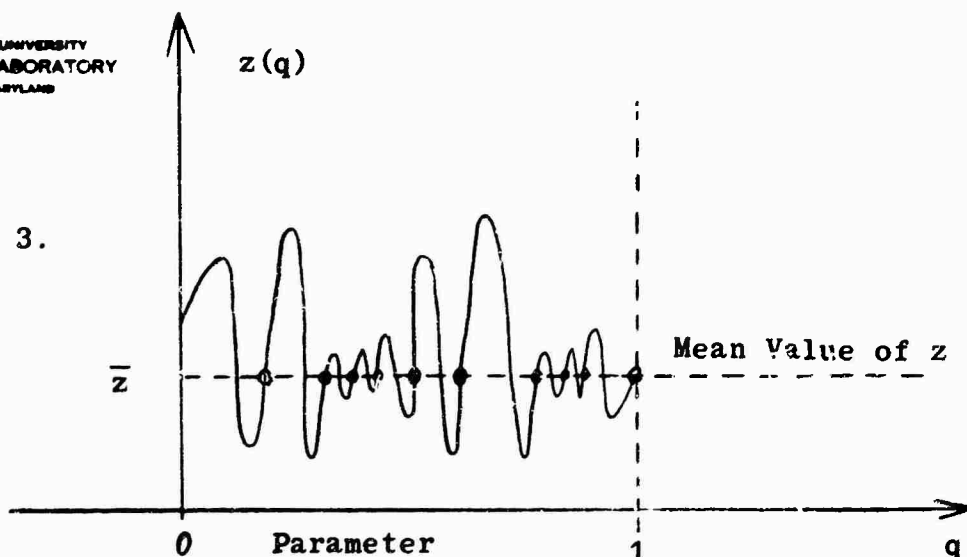
The mean count of lobes or the average rate of nulls was defined in Section A.II for a Gaussian, locally stationary, unbiased process by the average number of zero crossings with positive slope per unit of the independent parameter and given by Rice's formula (A.III.6). If the process is biased such as the uni-polar video signal of the target echo, the formula can still be used but must be applied to the unbiased process which is formed by subtracting the mean value from the biased process. (See Figure 3).

However in some cases (see Figure 4), eq. (A.III.6) will not give the count of lobes or nulls properly. If one is interested in the number of lobes or nulls in a strict sense, then one should count the maxima or minima of the process $z(q)$. Eq. (A.III.6) has to be replaced by another formula of Rice (cf. eq. (3.6-6) of Ref. [10]). One gets for the average number of maxima per unit of parameter q (say 1 sec or 1 rad or 1 Hertz) the term $\bar{M}_q(z;q)$:

$$\bar{M}_q(z;q) = \frac{1}{2\pi} \left\{ \frac{\rho_z^{(4)}(\tau;q)}{-\ddot{\rho}_z(\tau;q)} \right\}_{\tau=0}^{1/2} \quad (\text{A.V.11})$$

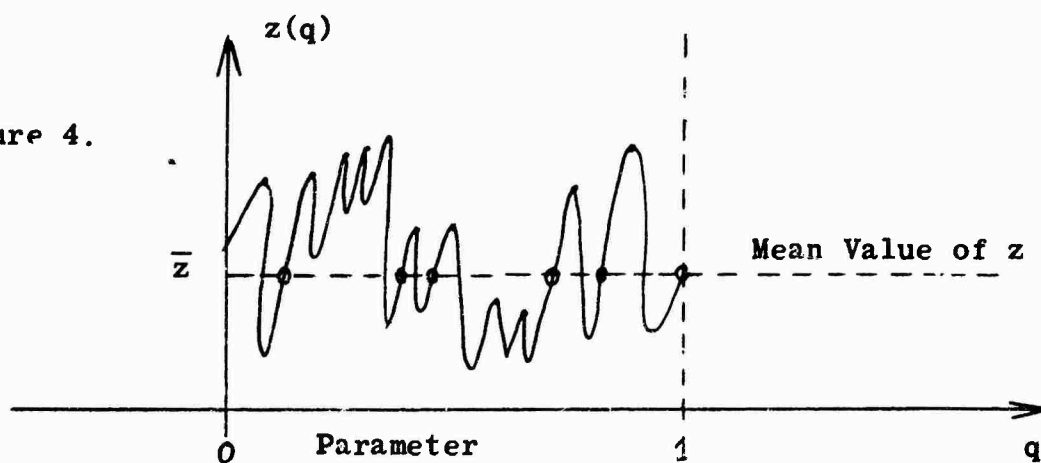
For specific targets (A.V.11) will lead to expressions similar in appearance as eqs. (10) to (12) of Section VII. However the mathematical and numerical computations will be slightly more complicated.

Figure 3.



Mean Number of Zero-Crossings per Unit of Parameter q of Unbiased Process $\Delta z = z - \bar{z}$. Here $\bar{N}_q(\Delta z; q) = 10$ (zeroes with positive slope).

Figure 4.



Mean Number of Maxima (Lobes) per Unit of q of Unbiased Process $\Delta z = z - \bar{z}$. Here $\bar{N}_q(\Delta z; q) = 6$ (zeros with positive slope) $\bar{M}_q(\Delta z; q) = 11$ (Maxima or Minima).

If the theory is applied to slender (one-dimensional) targets consisting of many, independent scatterers following a random, uniform distribution, eq. (A.V.11) yields a quantity \bar{M}_q which is roughly 50% greater than the quantity \bar{N}_q computed by eq. (10) of section VII

On the other hand, the latter result was based on a flat video-spectrum or a Dirac delta-function correlation (see eq. (A.III.5) of Appendix A-III). References [1], [21], and [22] showed that the video spectrum can be taken to be triangular (cf. eq. (A.V.5)). In many cases it is tapering off even more sharply. If our results are adjusted for this effect, one has to reduce \bar{M}_q by roughly 33%.

Combining these two adjustments for slender targets with many independent, uniformly distributed scatterers one obtains the following result: Eq. (10) of section VII yields an \bar{N}_q which in practice yields the count of maxima (lobes). This fact is born out by experimental measurements. Details of analytical, numerical and experimental results will be documented in future reports.

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13. ABSTRACT For slender, axially symmetrical targets that are much longer than the RF wavelength, radar scintillation is analyzed and described in terms of statistical parameters. The following sets of quantities were derived and computed: (a) mean radar cross sections, their RMS fluctuations and average lobing frequencies, (b) mean target centroids, their RMS deviations and average meandering rates These and other variables were expressed as functions of the aspect angle, radio frequency and of configurational details.		

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